Robust Portfolio Rebalancing with Transaction Cost Penalty – An Empirical Analysis

Abstract

The goal of this paper is to compare different techniques of reducing the sensitivity of optimal portfolios to uncertainty in expected return for a typical portfolio optimization problem. Specifically, we investigate whether including transaction costs into the optimization problem’s objective function addresses the robustness issue. We weigh this approach against the robust optimization method described in Goldfarb and Iyengar (2003). The latter directly incorporates the distribution of estimation errors in the optimization problem and determines the optimal portfolio allocation by selecting the “least” favorable realization of the expected returns in the investor’s uncertainty region.

Our analysis focuses on the return maximization problem with constraints on total risk or tracking error and a transaction cost penalty in the objective function. We demonstrate that not only are the effects of incorporating a transaction cost penalty into the optimization problem similar to those of modeling uncertainty in expected returns, there are also some interesting differences. We offer some insights into the observed interplay between modeling transaction costs and modeling return uncertainty.

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1. INTRODUCTION

The seminal paper by Harry Markowitz (1952) laid the foundation of modern quantitative finance theory and has shaped the quantitative investment process well into the 21st century. While elegant and well-rooted in economic theory, the original formulation of the return maximization problem can provide unintuitive results if used naively. One of the main reasons for this is the imprecision with which the input variables to the problem are typically estimated. Several authors provide specific examples of how prediction errors in the expected returns and/or in the covariance matrix can affect optimization results. Imprecision in forecasting covariances has been dealt with by using factor models, utilizing observed volatility clustering and, more recently, by adopting the concept of realized volatility.

Several techniques have been suggested to reduce the sensitivity of the optimal portfolios to uncertainty in expected returns, such as introducing weight constraints, Bayesian shrinkage and portfolio re-sampling, to name just a few. However, estimating expected returns remains a challenge. The haphazard use of return estimates often leads to so-called “error maximization”, i.e., the situation in which an optimizer allocates the largest weights to securities with the highest positive estimation error in alpha (defined as the difference between the estimated and true return). The problem can be so severe that a simple, equally-weighted portfolio can routinely outperform the Markowitz optimized portfolio.

A very effective, albeit indirect, way to mitigate the effect of estimation errors in expected returns is to incorporate a transaction cost penalty into the optimization problem's objective function. The robustness effect of doing so is achieved primarily as a consequence of two findings. First, extreme weights in individual stocks require more trading and, therefore, are often prevented by the presence of the transaction cost penalty. Second, the estimation error in expected returns is usually higher for stocks that are more expensive to trade. As a result, the transaction cost penalty is approximately proportional to the degree of uncertainty with respect to the stock’s expected alpha. Avoiding stocks that are more expensive to trade means avoiding stocks with greater uncertainty in expected return estimates.

Another way of countering uncertainties in the input variables is robust optimization. Instead of using point estimates, as in the classical Markowitz formulation, the robust optimization framework considers the distribution of estimation errors of returns directly in the optimization process. The true expected returns are defined via an uncertainty set that contains most of their possible realizations. This approach takes a conservative view and identifies the allocation of portfolio assets that have the best worst-case behavior.

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1 See, for instance, Jobson and Korkie (1981).
2 See Jagannathan and Ma (2004), Jorion (1986), Black and Litterman (1992), and Michaud (1989) for examples of the mentioned approaches.
3 DeMiguel, Garlappi and Uppal (2007) provide a good illustration of this problem.
4 Borkovec, Domowitz, Kiernan and Serbin (2009) illustrate the effects of incorporating transaction cost estimates into the mean-variance portfolio construction. The authors show that accounting for trading costs ex ante delivers portfolios that are more robust to noisy alpha signals, have superior net returns, broader diversification and lower turnover relative to standard mean-variance solutions. The robust effect of including a transaction cost penalty into the optimization problem has also been studied by Coppejans and Madhavan (2007), who show that including transaction cost estimates into the portfolio optimization problem mitigates extreme values of expected returns forecasts.
5 Goldfarb and Iyengar (2003) were among the first to apply the “best worst-case” formulation in order to model the input uncertainty in the portfolio optimization context. For a basic introduction of the robust optimization framework, also see pp. 308-319 in Fabozzi, Focardi and Kolm (2006).
The main goals of this paper are

a. to compare the out-of-sample performance of the robust solutions obtained by explicit inclusion of a transaction cost penalty in the objective function vs. modeling the uncertainty set around expected return estimates, and

b. to study the interplay between these two approaches when they are used simultaneously; in particular, to understand whether the joint usage can provide superior out-of-sample performance with regards to portfolio return and risk.

We demonstrate that the observed effects of incorporating transaction costs and modeling return uncertainties overlap to a large degree. For instance, we observe that as the forecasting ability of the money manager goes down, increasing the transaction cost penalty and the penalty for return uncertainty have similar effects. Both scenarios prevent the portfolio’s net return from deteriorating. In essence, our results suggest that if less skilled managers assume fund management responsibilities, they should trade less frequently and/or pick only stocks that are less risky. Our empirical results show that the relation between the optimal values for the transaction cost penalty and the penalty for return uncertainty is inverse and monotonic. At the same time, the paper demonstrates that there are some differences between the two approaches. There is a value added by modeling return uncertainty and simultaneously incorporating transaction costs at the portfolio formation stage.

Further, we contribute to the literature by offering some guidelines regarding the choice of the proper values of parameters characterizing the uncertainty region and the transaction cost penalty in order to maximize the out-of-sample net portfolio return. While it is a common practice to include transaction cost penalties and/or uncertainty regions around the estimated returns into the optimization problem, there is little guidance with respect to selecting the proper values of those parameters as functions of stock characteristics, manager’s forecasting ability, and other portfolio constraints that may be present in the problem.

In the current paper, we limit ourselves to the return maximization problem formulation with total risk and tracking error constraints for U.S. securities. The restriction to the U.S. securities is just technical, and we do not believe that our results will change qualitatively for other markets. Since we do not model the prediction errors in the forecasted covariance matrix, we confine our analysis to the return maximization problem. The analysis of other portfolio optimization formulations, such as minimizing the tracking error relative to an index, is left for future research.

This paper is organized as follows. Section 2 presents a mathematical formulation of the classical portfolio optimization problem and its robust counterpart. Section 3 contains out-of-sample results and associated discussion. Section 4 presents the paper’s conclusion.
2. Mathematical Formulation of Markowitz and Robust Optimization Approaches

2.1 Markowitz Formulation with Transaction Cost Penalty

A typical formulation of an optimal risk-constrained portfolio optimization problem with transaction cost penalty $\tau$ is:

$$\max_{\omega} \left( \hat{\mu}^T \omega - \tau \sum_{i=1}^{n} TC_i \left( K \cdot |\omega_i - \omega_{i,0}| \right) \right) / K$$

subject to

$$-0.2 \leq \omega_i \leq 0.2, \quad \sum_{i=1}^{n} \omega_i = 1$$

and

$$\omega^T \Sigma \omega \leq \sigma^2$$

in which

- $\omega = (\omega_1, \ldots, \omega_n)^T$ is the vector of portfolio weights,
- $\omega_0 = (\omega_{1,0}, \ldots, \omega_{n,0})^T$ is the vector of initial portfolio holdings,
- $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)^T$ is the vector of stock-specific expected returns,
- $n$ is the number of assets in the stock universe to be selected from,
- $TC_i \left( K \cdot |\omega_i - \omega_{i,0}| \right)$ is the total transaction costs (in dollars) of changing the allocation of asset $i$ from $K \cdot \omega_{i,0}$ to $K \cdot \omega_i$ dollars,
- $K$ is the initial portfolio value,
- $\Sigma$ is the covariance matrix of stock-specific returns,
- $\sigma$ is the total risk bound.
Constraint (2) forces the optimizer to allocate all the cash and bounds of the positions in individual securities (short or long) to 20% of the current portfolio wealth. Inequality (3) constrains the risk of the portfolio by $\sigma$.

Throughout this paper, the expected costs are derived from ITG’s Agency Cost Estimator (ACE)\(^6\). ACE is a dynamic structural econometric model, for the stock-specific expected price impact cost at the level of an individual order, differing by size of the order and the market conditions at the time of order submission. Permanent and transitory price impacts are explicitly modeled in such a way as to ensure that the first trade of a multi-trade order affects the prices of all subsequent sub-blocks sent to the market. These expected costs also depend on the trading strategy. We assume a 10% volume participation rate strategy throughout the paper.

The covariance estimates are provided by the monthly U.S. model from ITG’s suite of risk models. As is the case for the majority of such models, the market, sector, and industry factors capture differing sources of risk, augmented by growth and size factors on a per-stock basis. The factor covariance matrix is scaled using an option-implied adjustment coefficient to exploit the options market information with respect to future levels of risk. In contrast to some of the cross-sectional paradigms, the factor loadings are estimated in a time-series framework on a per-stock basis.\(^7\) The covariance matrix $\Sigma$ of stock returns is computed via the equation

$$\Sigma = V^T F V + D$$  \hspace{1cm} (4)

where

$F$ is the factor covariance matrix,

$V$ is the matrix of factor loadings, and

$D$ is the diagonal matrix of asset-specific variances.

### 2.2 Robust Formulation

The robust formulation of the portfolio optimization problem models the uncertainty in the input variables (e.g. expected returns) by specifying a set of values which contain the most likely possible realizations. There are several possible ways to describe these sets. However, most approaches reformulate the optimization problem as a “max-min problem”, where the expected utility function (e.g. portfolio return net of transaction costs) is maximized in the least optimal-case realization of the uncertain expected input variables.

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\(^7\) A complete description, including performance testing results, can be found in ITG’s Risk Models, Version 3, May 2008, available from the authors.
The uncertainty sets are typically modeled as confidence intervals around the parameters of interest. The probabilities can be determined on an asset-by-asset basis via the so-called “box uncertainty” region, or for all assets at once by the “ellipsoidal uncertainty” approach. Most approaches assume that the return distributions are multivariate normal. A non-parametric alternative is to use realized return frequencies, as described in Bienstock (2007). However, solving the resulting optimization problem in the non-parametric case is not straightforward and requires complicated solutions, while the majority of problems with parametric uncertainty sets can be relatively easily solved with standard conic algorithms.

In this paper, we use the ellipsoidal uncertainty sets, first implemented in the context of portfolio optimization by Goldfarb and Iyengar (2003). The ellipsoidal uncertainty defines the uncertainty region for all stocks at once and thus allows for possible dependencies between estimation errors.

The set of returns described by the ellipsoidal uncertainty region can be expressed in the form

\[ S^\kappa (\hat{\mu}) = \{ \mu | (\mu - \hat{\mu})^T \Sigma^{-1} (\mu - \hat{\mu}) < \kappa^2 \} \]  

(5)

where \( \hat{\mu} \) is the vector of expected returns, \( \kappa \) is the uncertainty aversion coefficient that defines how wide the uncertainty region is, and \( \Sigma^{-1} \) is the matrix of the estimation errors in expected returns \( \mu \). It is common to derive the uncertainty matrix \( \Sigma^{-1} \) from the covariance matrix \( \Sigma \), which is quite intuitive, since more volatile stocks have wider uncertainty regions associated with their expected returns. However, in general, the matrix \( \Sigma^{-1} \) might be completely independent of the covariance matrix \( \Sigma \), for example, if the vector of expected returns is obtained as a result of fundamental analysis. In our research, we assume that \( \Sigma^{-1} \propto \text{diag}(\Sigma) \). Making the uncertainty in expected return estimates proportional to the associated return volatilities is consistent with several fundamental principles of modern finance, such as the arbitrage pricing theory, which assumes that the expected return is strongly related to systematic risk.

Taking (4) into consideration, the robust counterparts of the optimization problem (1) can be described by the objective function

\[ \max_{\omega} \left( \hat{\mu}^T \omega - \tau \sum_{t=1}^n TC_t \left( K : |\omega_t - \omega_{t,0} | \right) \right) / K - \kappa \cdot \sqrt{\omega^T \Sigma^{-1} \omega}, \]  

(6)

with all remaining constraints (2) and (3) unchanged. The details of derivation can be found in Goldfarb and Iyengar (2003).

While having the parameters \( \kappa \) and \( \tau \) in the objective function is quite intuitive, it is not clear what values of those parameters would result in the highest out-of-sample performance in terms of net return and risk. Setting \( \kappa \) or \( \tau \) too low would potentially not allow for the full benefit of
robust optimization to be realized. Setting either of them too high will lead to overly conservative solutions that recommend almost no trading and, consequently, are likely to be suboptimal from a net return perspective.

In what follows, various choices of the trading cost aversion parameter $\tau$ and the uncertainty parameter $\kappa$ are considered and the corresponding out-of-sample results are presented. We also consider the case when constraint (3) is replaced with a corresponding tracking error constraint, i.e.,

$$
(\omega - \omega^*)^T \Sigma (\omega - \omega^*) \leq \delta^2
$$

(3')

where $\omega^*$ is the vector of benchmark weights, $\delta$ is the tracking error bound and the rest of the notation is as before.

Our empirical results show that selecting the “optimal” values for $\kappa$ and $\tau$ (in terms of out-of-sample return maximization) depends on the nature of the optimization problem (risk or tracking error-constrained), the money manager’s forecasting ability (proxied by the information coefficient IC), and, most importantly, the relation between $\kappa$ and $\tau$. We demonstrate that the relation between the optimal values of $\kappa$ and $\tau$ is inverse and monotonic. Varying either the uncertainty region or the transaction cost penalty by itself could mitigate noisy alpha signals, but using them together can lead to even better solutions.

3. Out-Of-Sample Test Results

Our out-of-sample testing will focus on the analysis of potential benefits of the robust formulation relative to the traditional mean-variance optimization problem.

Every month, starting in December 2003 and ending in December 2008, we form random portfolios of 100 stocks out of the eligible stock universe. Our universe is formed from all U.S. stocks with a market capitalization exceeding $64$ million (which is the approximate market capitalization cutoff for all Russell 2000 securities as of the time of writing this article), a trading price above $1$ and no more than 10 missing returns out of the last 60 trading days preceding each of the portfolio formation dates. We also require that the half-spread for each stock does not exceed the 95th percentile of all half-spreads for the Russell 3000 universe for each month of the out-of-sample period. The restriction on the spread costs helps to exclude extremely illiquid and expensive-to-trade stocks. The initial wealth of each of the portfolios is assumed to be $5$ million. After the random portfolio is formed, we run the optimization problems (1) and (6). We hold the optimized portfolio for one month, record its return and then perform optimization of the current portfolio for the next month. By the end of December 2008, we have a time series of 60 monthly portfolio returns. We repeat this exercise 25 times. In other words, we draw 25 random portfolios and aver-

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8 Obviously, this is an arbitrary choice of cutoff, which does not affect the qualitative nature of our results.

9 The $5M$ initial wealth corresponds to the average trading size of a portfolio of 100 assets with approximately 3-4% of the average daily volume for each stock.
age the out-of-sample statistics across time and across the portfolios. We also repeat this exercise for several values of parameters used to form the robust optimization problem. In what follows, we describe the computation of the inputs into the robust portfolio problem, explain the choice of parameters and discuss out-of-sample results.

In order to run the optimization problems (1) and (6), we need to simulate the “forecasted” excess return vector \( \hat{\mu} \) from the perspective of a typical money manager. We assume that the expected returns are based on a normally distributed random scheme, exploiting its information coefficient formulation,

\[
\hat{\mu}_{i,t} = IC \cdot \mu_{i,t}^{\text{observed}} + \left[ \sum_{t=1}^{T} \right] \cdot \sqrt{1-IC^2} \cdot \epsilon_i, \quad i = 1, \ldots, n, \tag{7}
\]

where \( \mu_{i,t} = R_{i,t} - r_f \), \( R_{i,t} \) is the realized stock return of security \( i \) in month \( t \), \( r_f \) is the risk-free rate, \( IC \) is the information coefficient (i.e. the proxy for the money manager’s forecasting ability) and \( \epsilon \sim N(0,1) \). We report the test results for the values of \( IC \) roughly corresponding to an average (IC=5%) and a poor forecasting ability (IC=2%).

In order to compare the performance of the portfolios obtained from the robust formulation with the results using the regular Markowitz framework, we introduce the parameter function \( P(\tau) \) defined as

\[
P_\kappa(\tau) = \max_{\kappa} R_{\text{robust}}(\tau, \kappa) - R_{\text{Markowitz}}(\tau) \tag{8}
\]

where \( R_{\text{Markowitz}}(\tau) \) and \( R_{\text{robust}}(\tau, \kappa) \) are out-of-sample returns of portfolios using the Markowitz and robust formulation, respectively, \( \kappa \) is the level of uncertainty and \( \tau \) is the trading cost aversion coefficient. The quantity in (8) can be interpreted as the relative performance of the robust portfolios over the Markowitz portfolios.

In order to explore the sensitivity of the problem with respect to the input parameters, we run the optimization for the input parameter values presented in Table 1.

| Table 1 |
|------------------|------------------|
| Robust uncertainty aversion coefficient \( \kappa \) | 1, 3, 5, 7, 9, 11, 14, 24 |
| Forecasting ability coefficient \( IC \) | 2%, 5% |
| Annualized risk constraint \( \sigma \) | 15%, 25%, 35% |
| Annualized tracking error constraint \( \delta \) | 3%, 5%, 8% |
| Trading cost aversion coefficient \( \tau \) | 0, 1, 5, 10, 30, 60 |
The values of $\kappa$ specified in Table 1 cover a wide range of confidence levels that define the ellipsoid region described by (5). The larger $\kappa$ is, the wider the uncertainty region becomes and, therefore, more unfavorable return outcomes can be produced. The value of the trading cost aversion coefficient $\tau$ varies from 0 (i.e., the optimization problem is run without a transaction cost penalty) to 60.

### 3.1. Average Forecasting Ability (IC = 5%)

The average out-of-sample monthly returns across portfolios and annualized risk are presented on the vertical axes in Figures 1 and 2, respectively. The values of the uncertainty aversion coefficient $\kappa$ are on the horizontal axis, with $\kappa = 0$ representing the optimal portfolios without robust formulation. We draw a separate curve for each value of the cost aversion coefficient $\tau$ specified in Table 1. Finally, we plot a separate chart for each value of the risk constraint $\sigma$ from Table 1.

**Figure 1: Monthly Out-Of-Sample Portfolio Return for Different Risk Constraints (IC=5%)**

**Figure 2: Annualized Out-Of-Sample Portfolio Risk for Different Risk Constraints (IC=5%)**
The return charts suggest that the robust formulation strictly dominates the traditional formulation across all cost aversion coefficients and for all three values of the total risk constraint. The degree of outperformance varies and the optimal value of $\kappa$ depends on $\tau$. Specifically, the optimal $\kappa$ is smaller for higher values of $\tau$. For instance, when the total risk constraint is 35%, setting $\tau$ to zero requires $\kappa \approx 7$ in order to maximize the out-of-sample net return, while with $\tau = 60$, the out-of-sample net return is maximized with $\kappa = 0$. The relationship between optimal $\kappa$ and $\tau$ is quite stable across different values of the total risk constraint. It also points to some overlap between increasing the cost aversion parameter and increasing the uncertainty aversion coefficient. However, the overlap is not complete, as the distance $\Delta$ (shown for $\tau = 10$) is greater than zero across all $\tau$ values (except perhaps when $\tau = 60$).

Figure 2 demonstrates that the annualized total risk of portfolios corresponding to the optimal $\kappa$ is, on average, always lower than the risk of portfolios that were not formed using the robust formulation. It is not clear, though, whether this result is related to the fact that we set $\Sigma \propto \text{diag}(\Sigma)$, or whether this observation is a more general result which holds even when the uncertainty matrix is unrelated to the asset covariance matrix.

As with the out-of-sample returns, increasing either the cost aversion or the uncertainty aversion coefficient has a similar effect on the portfolio risk (the risk decreases). For example, for IC=5% and annualized risk constraint of 35%, increasing $\tau$ from 0 to 60 while keeping $\kappa=0$ is equivalent, in terms of out-of-sample risk, to increasing $\kappa$ from 0 to 8 while keeping $\tau=0$.

### 3.2. Below Average Forecasting Ability (IC = 2%)

When the forecasting ability is below average (IC = 2%), the out-of-sample net returns decrease. However, the pattern of the relative performance of the robust formulation as defined in (8) remains similar to the one obtained with IC=5%. There are slight differences though: for example, as the forecasting ability decreases from 5% to 2%, the optimal coefficient of the cost aversion parameter $\tau$ increases from 10 to almost 60 (comparing Figure 1 with Figure 3 looking at the leftmost plot with the annualized risk constraint equaling 35% and $\kappa=0$). The conclusion is not new, but important enough to be repeated: “trading on imperfect information can be hazardous to one’s own wealth”.

**Figure 3: Monthly Out-Of-Sample Portfolio Return for Different Risk Constraints (IC=2%)**
Figures 3 and 4 above indicate that, in terms of out-of-sample net returns and risk, the robust formulation outperforms the classical one across a broad range of cost aversion coefficients and for every value of the total risk constraint considered. Moreover, the degree of outperformance, relative to the classical formulation, is higher than with IC=5%. We offer more discussion on this in the next section.

To shed more light on the interplay between using the robust formulation and the trading cost penalty, we present statistics on the average turnover for different values of $\tau$ and $\kappa$.

Figure 5 indicates that keeping $\tau$ at 0 and increasing $\kappa$ to 11 is approximately equivalent in terms of reducing turnover to increasing $\tau$ from 0 to 60 without implementing the robust feature. This robust effect of varying the transaction cost penalty is related to the fact that usually the stocks which are the most expensive to trade are also the most volatile. Therefore, increasing the penalty on the trading cost filters these stocks out from the optimal portfolio, thus contributing to its robustness.
3.3. More Discussion on $\kappa$ and the Interplay between $\kappa$ and $\tau$

Our empirical results show that the effects of incorporating transaction costs and modeling the return uncertainty overlap to a large degree. Nevertheless, there is value added in modeling return uncertainty on top of the use of a transaction cost penalty. Figure 6 summarizes the performance of the robust optimization formulation relative to the classical one, where the relative performance of the robust optimization is defined as in (8). For each value of $\tau$, only the best net return across the set of values of parameter $\kappa$ is presented. We also separate the results across the three values of the total risk constraint.

Figure 6: Relative Performance of Robust Portfolios for Different $\tau$ and Risk Constraints

The degree of outperformance of the robust formulation is related to the size of the feasibility region of the optimization problem, which in turn is inversely proportional to the strictness of the problem constraints. Figure 6 shows that the robust feature adds more incremental value to the portfolio optimization with total risk constraints when the trading cost penalty does not exist or when it is small. In both cases, the effects of the transaction cost penalty are not significant, thus making the uncertainty penalty an effective tool. More importantly, the robust formulation enhances the out-of-sample performance for any choice of the cost aversion parameter. We conclude that the effects of including transaction costs or estimation uncertainty of returns into the objective function are similar, but the overlap is not complete.

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The relative performance of the robust formulation is higher when the forecasting ability is low. Figure 6 indicates that the net return for the robust portfolio is ~5-25bp and 10-35bp higher than the net return for the classical portfolio with IC=5% and IC=2%, respectively (with $\tau$ fixed at 30). This result is intuitive: if portfolio managers are capable of achieving a high (positive) correlation between predicted and realized returns, they do not have to rely much on the robust formulation as a guard against estimation errors. Finally, if we tighten the risk constraints, the relative
performance of the robust optimization goes down as the size of the investment opportunity set decreases.

As Figures 1-5 show, increasing both the cost aversion coefficient and the uncertainty penalty enhances the robustness of the optimal portfolio. The optimal $\kappa$ (the value which delivers the highest out-of-sample net return) is inversely related to the level of $\tau$. In other words, we do not need a high level of $\kappa$ to maximize the return when $\tau$ is sufficiently high. Figure 7 below highlights in more detail the inverse relation between the return-maximizing values of $\tau$ and $\kappa$. For each choice of the parameters $\sigma$, IC and $\tau$, we plot the value of $\kappa$ that maximizes the out-of-sample net return. The optimal level of the uncertainty penalty inversely depends on the level of IC. For example, by fixing $\tau$ at 30, and $\sigma$ at 35% we observe that the levels of $\kappa$ which correspond to the highest out-of-sample net return are $\kappa=3$ for IC=5% and $\kappa=5$ for IC=2%. The points in green reflect the optimal pairs of $\tau$ and $\kappa$ that yield the highest out-of-sample return for the given risk constraint and IC scenario.

**Figure 7: Optimal Return-maximizing Value of $\kappa$ For Different Total Risk Constraints, IC Values and Trading Cost Penalties**

As discussed above, as $\tau$ increases, the optimal value of $\kappa$ that is necessary to maximize out-of-sample net return decreases. It is consistent with the notion that both $\tau$ and $\kappa$ can be used to mitigate noisy alpha signals. Furthermore, Figure 7 suggests that the optimal trading cost penalty $\tau^*$ marked by the green dots in the chart (denoting the $\tau$ values of the unique pairs ($\tau$, $\kappa$) that yields the highest out-of-sample return for a given scenario) is decreasing in IC and stays approximately constant for different total risk constraints.
We also note that as the risk constraint $\sigma$ is tightened, the optimal level of $\kappa$ increases. For example, tightening the total risk constraint from 35% to 15% requires a slight increase in the optimal level of $\kappa$ from 3 to 5 (for IC=5% and $\tau=30$, although a qualitatively similar behavior is detected for other parameter choices). This result might be surprising at first glance, but can be explained by the following reasoning. The optimization problem in equation (6) with constraint (3) is equivalent to

$$\max_{\omega} \left( \hat{\mu}^T \omega - \tau \sum_{i=1}^{n} TC_i \left( K \cdot [\omega_j - \omega_{j,0}] \right) \right) \left( K - \kappa \cdot \sqrt{\omega^T \Sigma \omega - \lambda \cdot \omega^T \Sigma \omega} \right)$$

where $\lambda = \lambda (\sigma)$ is the Lagrange multiplier for the risk constraint $\sigma$. In particular, tighter values of the risk constraint $\sigma$ are associated with higher values of $\lambda$, which stems from the fact that violations of the total risk constraint become more costly. Therefore, as the risk constraint (3) becomes tighter, the above optimization problem becomes less and less centered on the uncertainty in the forecast of the alphas and more on the portfolio variance, which to a large extent is determined by off-diagonal elements of the covariance matrix. Robust optimization, on the other hand, focuses on the diagonal elements of the covariance matrix. This implies that the robust optimization effect starts to be dominated by the risk constraint as the latter becomes stricter. Therefore, to compensate for performance slippage in terms of returns, resulting from increased emphasis on risk, one should increase the value of uncertainty penalty $\kappa$ to put emphasis back on the robustness of expected return estimates.

3.4. Out-Of-Sample Results with Tracking Error Constraints

In what follows, we briefly summarize our out-of-sample results when substituting the total risk constraint (3) with the tracking error constraint (3'). The associated robust problem formulation might be relevant to active/enhanced indexers and, therefore, it is of practical interest to verify if any conclusions from the previous sections need to be changed.

We use the same parameter values as in Table 1, except that we now consider the thresholds of 3%, 5% and 8% for the tracking error constraint. We pick the S&P 500 as a benchmark, and to be consistent with the industry practice, we narrow our initial universe of eligible names to the S&P 500 constituents. The sample period and simulation procedure is kept unchanged from the previous section.

The results are qualitatively the same as with the total risk constrained problem, and, therefore, we keep this discussion brief. The robust optimal portfolios continue to dominate the Markowitz portfolios in terms of out-of-sample net return when the tracking error constraint is used in place of the total risk constraint. The dependence of the robust performance on the different levels of tracking error constraints is similar to the one observed for the total risk constraint. As the constraint becomes tighter, the degree of outperformance goes down. Under a very binding 3% tracking error constraint, the robust optimizer is able to offer only a very insignificant improvement over the classical formulation. There is simply not enough room to select sufficiently different stocks.
Similar to the total risk constraint, the optimal level of the uncertainty penalty $\kappa$ is higher as IC goes down and the outperformance is higher for lower levels of IC. The optimal value of $\tau$ does not depend on the tracking error constraint but instead depends on the money manager’s forecasting ability: for instance, the optimal value of $\tau$ increases from 10 to 30 when IC goes down from 5% to 2% for an 8% tracking error constraint. Finally, the effects of changing $\tau$ and $\kappa$ on portfolio turnover presented in Figure 5 remain qualitatively similar for the tracking error constrained problem.

The relation between the return-maximizing cost aversion penalty $\tau$ and the uncertainty penalty $\kappa$ for the tracking error constrained problem exhibits the same pattern as for the total risk constrained problem. It is depicted in Figure 8. Similar to the total risk constrained problem, there are three major observations to note. First, the optimal $\kappa$ and $\tau$ are decreasing in IC, i.e., a lower forecasting ability requires higher levels of penalties in order to maximize out-of-sample net return. Second, the values of $\kappa$ and $\tau$ need to be increased to achieve good out-of-sample results as the tracking error constraint becomes stricter. Third, the optimal values of $\kappa$ and $\tau$ are inversely proportional to each other. For example, $\kappa$ needs to be set to 8 in order to maximize the net return when $\tau=1$; but setting $\kappa=5$ is sufficient when $\tau=60$ (for IC=5% and tracking error constraint of 8%). Finally, as before, the points in green denote the pairs of $\kappa$ and $\tau$ resulting in the highest out-of-sample portfolio net return. Additional tables and charts are available upon request.

Figure 8: Optimal Return-Maximizing Value of $\kappa$ For Different Tracking Error (TE) Risk Constraints, IC Values and Trading Cost Penalties
4. Conclusion

In this paper we conduct a comprehensive analysis of the effects of incorporating both a robustness constraint and transaction cost penalty into the typical portfolio return maximization problem. Our results suggest that the transaction cost penalty is very effective in addressing estimation errors in expected returns, as the resulting optimal portfolios dominate the classical mean variance portfolios for a wide range of trading cost penalties, risk and tracking error constraints. The results also suggest that the benefits of the robust formulation become more pronounced when the quality of the alpha signals deteriorates. We analyze the joint effect of including transaction cost and uncertainty penalties into the optimization problem. While these effects overlap to a large degree, there is a value added in including both types of penalties at the portfolio formation stage.

Most of our results are qualitative in nature and, therefore, can be used as an initial guide for understanding the effect and the impact of the uncertainty and transaction cost penalties on the robust solution of the Markowitz problem. However, it should be understood that the findings in this paper are not meant to be used as a blanket recommendation for general portfolio optimization problems, as the optimal parameter values affecting robustness depend on the problem type, money manager’s forecasting ability and other parameters present in the problem.

Our most interesting results can be summarized as follows:

- The optimal value of the uncertainty penalty $\kappa$ should be chosen in conjunction with the transaction cost penalty $\tau$. Our results indicate that the optimal $\kappa$ (the value that achieves the highest out-of-sample net return) is inversely related to the level of $\tau$. Although there is some overlap between the two effects, the robust formulation can enhance the out-of-sample net return performance.

- The optimal value of the uncertainty penalty depends on the forecasting ability of a portfolio manager: higher skilled managers are better off choosing small penalties.

- The optimal value of the uncertainty penalty is proportional to the tightness of the portfolio risk constraint. This result is independent of whether a total risk or a tracking error constraint is used. However, since the sensitivity of $\kappa$ with respect to the risk constraint seems to be quite low, we conclude that using slightly suboptimal values for the uncertainty penalty is not likely to have a big impact on the portfolio performance.

One of the key assumptions in this paper is that the matrix encapsulating the uncertainty of the expected returns is proportional to the return covariance matrix. While this assumption is intuitive and can be justified, it would be interesting to see how the test results change if it is relaxed. Finally, since the optimal coefficients for the transaction cost and the uncertainty penalties are related to each other, it would be interesting to decompose the problem and analyze separately the optimal level of $\tau$ for alternative scenarios.
References


